An independent parametrization of an acute triangle and its applications

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Let us start with a problem that was proposed at the **40^{th} IMO Vietnam Team Selection Test, 2001 (Problem 1, day 2)**.

P1. Let $a, b, c > 0$ and $p, q, r > 0$. Determine the minimum value of $\frac{p}{x} + \frac{q}{y} + \frac{r}{z}$, where x, y, z are positive real numbers such that $ayz + bzx + cxy \leq d$. (In the original problem $p = 1, q = 2, r = 3, a = 2, b = 8, c = 21, d = 12$).

In what follows we will prove that any three positive numbers can be considered as distances from the circumcenter of an acute triangle to its sides (exact trilinear coordinates of the circumcenter). They will be defined in a certain way as shown in part **i.** of **Theorem 1** below. In addition, we will show that this triangle has a distinct feature described in part **ii.** of **Theorem 1** and can be written as inequality **(5).**

Theorem 1. *Let k, l and m be positive real numbers, then*

i. There is a unique acute triangle for which these numbers are the distances from its circumcenter to its sides and the sidelengths of the triangle are

$$a = 2\sqrt{R^2 - k^2}, b = 2\sqrt{R^2 - l^2}, c = 2\sqrt{R^2 - m^2}$$

where R (circumradius) is the only positive root of the cubic equation

$$t^3 - t\left(k^2 + l^2 + m^2\right) - 2klm = 0;$$

ii. The area, F, of this triangle is

$$\min_{\alpha,\beta,\gamma \in \left(0, \frac{\pi}{2}\right)} \left(k^2 \tan\alpha + l^2 \tan\beta + m^2 \tan\gamma\right) \text{ where, } \alpha + \beta + \gamma = \pi.$$

Proof. i. *Necessity.* Let ABC be an acute triangle and let k, l, m be the distances from its circumcenter O to the sides BC, CA, AB, respectively. Let R be the circumradius of ABC. Because $OA = OB = OC = R$ and $\angle BOC = 2A$, $\angle COA = 2B, \angle AOB = 2C$, we have

$$\frac{k}{R} = \cos A, \frac{l}{R} = \cos B, \frac{m}{R} = \cos C.$$

Using the identity $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1$, which holds for any $\alpha,\beta,\gamma>0$ with $\alpha+\beta+\gamma=\pi$, we obtain

$$\frac{k^2}{R^2}+\frac{l^2}{R^2}+\frac{m^2}{R^2}+\frac{2klm}{R^3}=1,$$

or equivalently

$$R^3-R\left(k^2+l^2+m^2\right)-2klm=0. \tag{1}$$

Note that R is the only positive root of the cubic equation (1)

$$t^3-t\left(k^2+l^2+m^2\right)-2klm=0$$

because the function $\varphi(t)=1-\dfrac{2klm}{t^3}-\dfrac{k^2+l^2+m^2}{t^2}$ is increasing on the interval $(0,\infty)$.

Sufficiency. Let k,l,m be positive real numbers and consider the cubic equation (1). Note that

$$\varphi\left(\sqrt{k^2+l^2+m^2}\right)<0 \text{ as } \lim_{t\to\infty}\varphi(t)=1$$

and $\varphi(t)$ is increasing on the interval $(0,\infty)$. Hence equation (1) has a single positive real root on $\left(\sqrt{k^2+l^2+m^2},\infty\right)$, which we denote by R. Because

$$R^3-R\left(k^2+l^2+m^2\right)-2klm=0 \iff \frac{k^2}{R^2}+\frac{l^2}{R^2}+\frac{m^2}{R^2}+\frac{2klm}{R^3}=1$$

and $\dfrac{k}{R},\dfrac{l}{R},\dfrac{m}{R}<1$ for angles

$$\alpha_1=\cos^{-1}\left(\frac{k}{R}\right),\beta_1=\cos^{-1}\left(\frac{l}{R}\right),\gamma_1=\cos^{-1}\left(\frac{m}{R}\right)$$

we have

$$\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1+2\cos\alpha_1\cos\beta_1\cos\gamma_1=1 \tag{2}$$

and $\alpha_1,\beta_1,\gamma_1\in\left(0,\dfrac{\pi}{2}\right)$. It follows that $\alpha_1+\beta_1+\gamma_1=\pi$. Indeed, since

$$\begin{aligned}
&\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1+2\cos\alpha_1\cos\beta_1\cos\gamma_1-1\\
&\quad=\cos^2\gamma_1+\frac{\cos 2\alpha_1+\cos 2\beta_1}{2}+\cos\gamma_1\left(\cos(\alpha_1+\beta_1)+\cos(\alpha_1-\beta_1)\right)\\
&\quad=\cos^2\gamma_1+\cos(\alpha_1+\beta_1)\cos(\alpha_1-\beta_1)\\
&\quad+\cos\gamma_1\cos(\alpha_1+\beta_1)+\cos\gamma_1\cos(\alpha_1-\beta_1)\\
&\quad=\left(\cos\gamma_1+\cos(\alpha_1+\beta_1)\right)\left(\cos\gamma_1+\cos(\alpha_1-\beta_1)\right)\\
&\quad=4\cos\varphi\cos(\varphi-\alpha_1)\cos(\varphi-\beta_1)\cos(\varphi-\gamma_1)
\end{aligned}$$

where $\varphi = \dfrac{\alpha_1 + \beta_1 + \gamma_1}{2}$ and $\varphi - \alpha_1, \varphi - \beta_1, \varphi - \gamma_1 \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$, $\varphi \in \left(0, \dfrac{3\pi}{4}\right)$. Then equation (2) is equivalent to

$$\cos\varphi \cos(\varphi - \alpha_1)\cos(\varphi - \beta_1)\cos(\varphi - \gamma_1) = 0 \iff \cos\varphi = 0 \iff \varphi = \frac{\pi}{2}.$$

Thus we can conclude that R and $\alpha_1, \beta_1, \gamma_1$ determine an acute triangle ABC with sides

$$BC = 2R\sin\alpha_1 = 2\sqrt{R^2 - k^2}$$
$$CA = 2R\sin\beta_1 = 2\sqrt{R^2 - l^2}$$
$$AB = 2R\sin\gamma_1 = 2\sqrt{R^2 - m^2}$$

and circumradius R such that

$$R\cos\alpha_1 = k$$
$$R\cos\beta_1 = l$$
$$R\cos\gamma_1 = m$$

are distances from the circumcenter to the sides BC, CA, and AB, respectively.

ii. First, we will prove that for any $\alpha, \beta \in \left(0, \dfrac{\pi}{2}\right)$ the following inequality holds

$$k^2\tan\alpha + l^2\tan\beta \geq \frac{2kl}{\sin(\alpha+\beta)} - \left(k^2 + l^2\right)\cot(\alpha+\beta) \qquad (3).$$

Because $\cos\alpha, \cos\beta, \sin(\alpha+\beta) > 0$, we obtain

$$\begin{aligned}
(3) &= k^2\left(\tan\alpha + \cot(\alpha+\beta)\right) + l^2\left(\tan\beta + \cot(\alpha+\beta)\right) \geq \frac{2kl}{\sin(\alpha+\beta)} \\
&\iff \frac{k^2\cos\beta}{\cos\alpha\sin(\alpha+\beta)} + \frac{l^2\cos\alpha}{\cos\beta\sin(\alpha+\beta)} \geq \frac{2kl}{\sin(\alpha+\beta)} \\
&\iff (k\cos\beta - l\cos\alpha)^2 \geq 0.
\end{aligned}$$

Equality occurs when $\dfrac{k}{\cos\alpha} = \dfrac{l}{\cos\beta}$. Let $\alpha, \beta, \gamma \in \left(0, \dfrac{\pi}{2}\right)$ and $\alpha + \beta + \gamma = \pi$ then using inequality **(3)** we obtain

$$\begin{aligned}
k^2\tan\alpha + l^2\tan\beta + m^2\tan\gamma &\geq \frac{2kl}{\sin(\alpha+\beta)} - \left(k^2 + l^2\right)\cot(\alpha+\beta) + m^2\tan\gamma \\
&= h(\gamma),
\end{aligned}$$

where $h(\gamma) = \dfrac{2kl}{\sin\gamma} + \left(k^2 + l^2\right)\cot\gamma + m^2\tan\gamma$. Because $\cos\beta = -\cos(\alpha + \lambda)$, then the equality in the above inequality can be written as

$$\begin{aligned}
\frac{k}{\cos\alpha} = \frac{l}{\cos\beta} &\iff k\cos\beta = l\cos\alpha \\
&\iff k\left(-\cos\lambda\cos\alpha + \sin\gamma\sin\alpha\right) = l\cos\alpha \\
&\iff -k\cos\gamma + k\tan\alpha\sin\gamma = l \\
&\iff k\tan\alpha = \frac{l}{\sin\gamma} + k\cot\gamma \\
&\iff k^2\tan^2\alpha = \frac{l^2 + k^2 + 2kl\cos\gamma}{\sin^2\gamma} - k^2 \\
&\iff \frac{k^2}{\cos^2\alpha} = \frac{l^2 + k^2 + 2kl\cos\gamma}{\sin^2\gamma}.
\end{aligned}$$

Thus the equality case is $\dfrac{k}{\cos\alpha} = \dfrac{l}{\cos\beta} = \dfrac{\sqrt{l^2 + k^2 + 2kl\cos\gamma}}{\sin\gamma}$.

Now we will find the minimum of $h(\gamma)$ on the interval $\left(0, \dfrac{\pi}{2}\right)$. We have

$$h'(\gamma) = -\frac{2kl\cos\gamma}{\sin^2\gamma} - \frac{k^2 + l^2}{\sin^2\gamma} + \frac{m^2}{\cos^2\gamma} = \frac{\cos\gamma}{m\sin^2\gamma} \cdot P\left(\frac{m}{\cos\gamma}\right),$$

where $P(t) = t^3 - t\left(k^2 + l^2 + m^2\right) - 2klm$. Equation $P(x) = 0$ has a single positive root $t = R$. Furthermore,

$$P(t) < 0 \iff \varphi(t) < \varphi(R) = 0 \text{ for } t \in (0, R)$$

and

$$P(t) > 0 \iff \varphi(t) > \varphi(R) = 0 \text{ for } t \in (R, \infty)$$

hence the local minimum of $h(\gamma)$, attained at $\gamma = \gamma_1 = \cos^{-1}\left(\dfrac{m}{R}\right)$, is also a global minimum. The lower bound of $h(\gamma_1)$ for $k^2\tan\alpha + l^2\tan\beta + m^2\tan\gamma$, can be reached when both of the inequalities $k^2\tan\alpha + l^2\tan\beta + m^2\tan\gamma \geq h(\gamma) \geq h(\gamma_1)$ are in fact equalities, i.e. if and only if

$$\left\{\begin{array}{c} \dfrac{k}{\cos\alpha} = \dfrac{l}{\cos\beta} = \dfrac{\sqrt{l^2 + k^2 + 2kl\cos\gamma}}{\sin\gamma} \\ \gamma = \gamma_1 \end{array}\right. . \tag{4}$$

Because

$$\begin{aligned} l^2+k^2+2kl\cos\gamma_1 &= l^2+k^2+\frac{2klm}{R} \\ &= \frac{R\left(l^2+k^2\right)+2klm}{R} = \frac{R^3-Rm^2}{R} = R^2-m^2 \\ &= R^2\sin^2\gamma_1 \end{aligned}$$

, $\dfrac{\sqrt{l^2+k^2+2kl\cos\gamma_1}}{\sin\gamma_1} = R$ and, therefore,

$$(4) \iff \begin{cases} \dfrac{k}{\cos\alpha} = \dfrac{l}{\cos\beta} = R \\ \gamma = \gamma_1 \end{cases} \iff \begin{cases} \cos\alpha = \dfrac{k}{R} \\ \cos\beta = \dfrac{l}{R} \\ \gamma = \gamma_1 \end{cases}$$

$$\iff \begin{cases} \cos\alpha = \cos\alpha_1 \\ \cos\beta = \cos\beta_1 \\ \gamma = \gamma_1 \end{cases} \iff \alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1.$$

Thus

$$\begin{aligned} &\min\left\{k^2\tan\alpha + l^2\tan\beta + m^2\tan\gamma : \ 0<\alpha,\beta,\gamma<\frac{\pi}{2},\ \alpha+\beta+\gamma=\pi\right\} \\ &= k^2\tan\alpha_1 + l^2\tan\beta_1 + m^2\tan\gamma_1, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= \cos^{-1}\left(\frac{k}{R}\right) \\ \beta_1 &= \cos^{-1}\left(\frac{l}{R}\right) \\ \gamma_1 &= \cos^{-1}\left(\frac{m}{R}\right) \\ \iff \ \tan\alpha_1 &= \frac{\sqrt{R^2-k^2}}{k} \\ \tan\beta_1 &= \frac{\sqrt{R^2-l^2}}{l} \\ \tan\gamma_1 &= \frac{\sqrt{R^2-m^2}}{m}, \end{aligned}$$

and R is the single positive root of equation $t^3 - t\left(k^2+l^2+m^2\right) - 2klm = 0$. Because

$$\begin{aligned} k^2\tan\alpha_1 + l^2\tan\beta_1 + m^2\tan\gamma_1 &= k\sqrt{R^2-k^2} + l\sqrt{R^2-l^2} + m\sqrt{R^2-m^2} \\ &= F, \end{aligned}$$

where F is the area of an acute triangle defined by circumradius R and distances k, l, m from circumcenter to the sides,

$$\min\left\{k^2\tan\alpha+l^2\tan\beta+m^2\tan\gamma:\ 0<\alpha,\beta,\gamma<\frac{\pi}{2},\ \alpha+\beta+\gamma=\pi\right\}=F,$$

$\square$

Remark 1. The result in part **ii.** of **Theorem 1** can be represented in an inequality as

$$k^2\tan\alpha+l^2\tan\beta+m^2\tan\gamma\geq k\sqrt{R^2-k^2}+l\sqrt{R^2-l^2}+m\sqrt{R^2-m^2}, \qquad (5)$$

where $0<\alpha,\beta,\gamma<\dfrac{\pi}{2},\ \alpha+\beta+\gamma=\pi$. Equality occurs if and only if

$$\tan\alpha=\frac{\sqrt{R^2-k^2}}{k},\ \tan\beta=\frac{\sqrt{R^2-l^2}}{l},\ \tan\gamma=\frac{\sqrt{R^2-m^2}}{m}.$$

If we let $u=\cot\alpha, v=\cot\beta, w=\cot\gamma$, then (5) becomes

$$\frac{k^2}{u}+\frac{l^2}{v}+\frac{m^2}{w}\geq k\sqrt{R^2-k^2}+l\sqrt{R^2-l^2}+m\sqrt{R^2-m^2}, \qquad (6)$$

where $u, v, w>0$ and $uv+vw+wu=1$. Equality occurs if and only if

$$u=\frac{k}{\sqrt{R^2-k^2}}, v=\frac{l}{\sqrt{R^2-l^2}}, w=\frac{m}{\sqrt{R^2-m^2}}.$$

Remark 2. Let $\Delta(x,y,z)=2xy+2yz+2zx-x^2-y^2-z^2$. Because $a=2\sqrt{R^2-k^2}, b=2\sqrt{R^2-l^2}, c=2\sqrt{R^2-m^2}$ and $16F^2=\Delta\left(a^2,b^2,c^2\right)$,

$$\begin{aligned}F^2&=2\sum_{cyc}\left(R^2-l^2\right)\left(R^2-m^2\right)-\sum_{cyc}\left(R^2-k^2\right)^2\\&=R^2\left(k^2+l^2+m^2\right)+6Rklm+\Delta\left(k^2,l^2,m^2\right)\end{aligned}$$

and

$$\begin{aligned}F&=k\sqrt{R^2-k^2}+l\sqrt{R^2-l^2}+m\sqrt{R^2-m^2}\\&=\sqrt{R^2\left(k^2+l^2+m^2\right)+6Rklm+\Delta\left(k^2,l^2,m^2\right)}.\end{aligned}$$

Application 1. We will use the above theorem to solve problem **P1.** Note that the minimum of $\dfrac{p}{x}+\dfrac{q}{y}+\dfrac{r}{z}$ cannot be attained at (x,y,z) for which $ayz+bzx+cxy<d$. Actually, if $ayz+bzx+cxy<d$ then for $z_1=\dfrac{d-cxy}{ay+bx}$

$$\frac{p}{x}+\frac{q}{y}+\frac{r}{z_1}<\frac{p}{x}+\frac{q}{y}+\frac{r}{z},\quad \text{because } z<z_1.$$

Thus $\min\left\{\frac{p}{x}+\frac{q}{y}+\frac{r}{z} \;\mid\; x,y,z>0 \text{ and } ayz+bzx+cxy\le d\right\} =$

$$\min\left\{\frac{p}{x}+\frac{q}{y}+\frac{r}{z} \;\mid\; x,y,z>0 \text{ and } ayz+bzx+cxy = d\right\}.$$

Let $u = x\sqrt{\frac{bc}{ad}}, v = y\sqrt{\frac{ca}{bd}}$ and $w = z\sqrt{\frac{ab}{cd}}$. Then $x = u\sqrt{\frac{ad}{bc}}$, $y = v\sqrt{\frac{bd}{ca}},\; z = w\sqrt{\frac{cd}{ab}}$, where u, v, w are any positive real numbers. Then

$$ayz+bzx+cxy = d \iff \frac{a}{d}yz+\frac{b}{d}zx+\frac{c}{d}xy = 1 \iff uv+vw+wu = 1$$

and

$\frac{p}{x}+\frac{q}{y}+\frac{r}{z} = \frac{k^2}{u}+\frac{l^2}{v}+\frac{m^2}{w}$, where $k^2 = p\sqrt{\frac{bc}{ad}}, l^2 = q\sqrt{\frac{ca}{bd}},\; m^2 = r\sqrt{\frac{ab}{cd}}$. Thus we obtain following equivalent representation of our problem: Find the min $\left\{\frac{k^2}{u}+\frac{l^2}{v}+\frac{m^2}{w} \;\mid u, v, w > 0 \text{ and } uv +\right.$

Using part **ii.** of **Theorem 1** we obtain

$$\begin{aligned} &min\left\{\frac{k^2}{u}+\frac{l^2}{v}+\frac{m^2}{w} \;\mid u,v,w>0 \text{ and } uv+vw+wu=1\right\} = \frac{k^2}{u_1}+\frac{l^2}{v_1}+\frac{m^2}{w_1} \\ &= k\sqrt{R^2-k^2}+l\sqrt{R^2-l^2}+m\sqrt{R^2-m^2}, \end{aligned}$$

where $u_1 = \frac{k}{\sqrt{R^2-k^2}},\; v_1 = \frac{l}{\sqrt{R^2-l^2}}, w_1 = \frac{m}{\sqrt{R^2-m^2}}$. Then for the initial problem we have

$$\begin{aligned} &\min\left\{\frac{p}{x}+\frac{q}{y}+\frac{r}{z} \;:\; x,y,z>0 \;\text{ and } ayz+bzx+cxy\le d\right\} = \frac{p}{x_1}+\frac{q}{y_1}+\frac{r}{z_1} \\ &= k\sqrt{R^2-k^2}+l\sqrt{R^2-l^2}+m\sqrt{R^2-m^2}, \end{aligned}$$

where

$$k^2 = p\sqrt{\frac{bc}{ad}}$$
$$l^2 = q\sqrt{\frac{ca}{bd}}$$
$$m^2 = r\sqrt{\frac{ab}{cd}}$$
$$x_1 = \frac{p}{k\sqrt{R^2-k^2}}$$
$$y_1 = \frac{q}{k\sqrt{R^2-l^2}}$$
$$z_1 = \frac{r}{m\sqrt{R^2-m^2}}$$

and R is a positive root of the cubic equation (1).

Application 2. Due to the well known duality orthocenter↔circumcenter, **Theorem 1** can be represented in another equivalent form:

Theorem 2. *Let k, l, m be arbitrary real positive numbers. Then*

i. *There is a unique acute triangle, for which these numbers are the distances from its orthocenter to its verteces and its sidelengths are*

$$a = \sqrt{4R^2 - k^2}, b = \sqrt{4R^2 - l^2}, c = \sqrt{4R^2 - m^2},$$

where R (circumradius) is the only positive root of a cubic equation

$$4t^3 - t\left(k^2 + l^2 + m^2\right) - klm = 0.$$

ii. *The following equality holds*

$$\min\left\{k^2 \tan\alpha + l^2 \tan\beta + m^2 \tan\gamma : \alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right) \text{ and } \alpha + \beta + \gamma = \pi\right\} = 4F,$$

where F is area of this triangle.

Proof. Numbers k, l, m, the distances from the orthocenter H of triangle ABC to its verteces, can at the same time be considered as distances from circumcenter $O_1 = H$ of triangle $A_1B_1C_1 \cong 2ABC$. Thus if R_1 is a single positive root of $t^3 - t\left(k^2 + l^2 + m^2\right) - 2klm = 0$ and

$$\alpha_1 = \cos^{-1}\left(\frac{k}{R_1}\right)$$
$$\beta_1 = \cos^{-1}\left(\frac{l}{R_1}\right)$$
$$\gamma_1 = \cos^{-1}\left(\frac{m}{R_1}\right)$$

then for $R = \dfrac{R_1}{2}$ we have

$$\alpha_1 = \cos^{-1}\left(\frac{k}{2R}\right)$$
$$\beta_1 = \cos^{-1}\left(\frac{l}{2R}\right)$$
$$\gamma_1 = \cos^{-1}\left(\frac{m}{2R}\right)$$

and R is single positive root of equation

$$4t^3 - t\left(k^2 + l^2 + m^2\right) - klm = 0.$$

Let $A_1B_1C_1$ be an acute triangle defined by angles $\alpha_1, \beta_1, \gamma_1$ and R_1 as circumradius and let $a_1 = B_1C_1, b_1 = C_1A_1, c_1 = A_1B_1$. Then according to **Theorem 1**, we have

$$a_1 = 2\sqrt{R_1^2 - k^2}, b_1 = 2\sqrt{R_1^2 - l^2}, c_1 = 2\sqrt{R_1^2 - m^2}$$

and

$$\min\left\{k^2\tan\alpha + l^2\tan\beta + m^2\tan\gamma \vdots 0 < \alpha, \beta, \gamma < \frac{\pi}{2},\ \alpha + \beta + \gamma = \pi\right\}$$
$$= [A_1B_1C_1] = 4F,$$

where F is area of triangle ABC determined by angles $\alpha_1, \beta_1, \gamma_1$ and $R = \dfrac{R_1}{2}$ with sides $a = \dfrac{a_1}{2} = \sqrt{4R^2 - k^2}, b = \dfrac{b_1}{2} = \sqrt{4R^2 - l^2}, c = \dfrac{c_1}{2} = \sqrt{4R^2 - m^2}$. □

[1] **CRUX, Vol.30, No1, February, 2004, p.17.**